Introduction to Abstract Algebra

with Applications to Social Systems

Women in Mathematics

SWIM

Summer Workshop in Mathematics

FOR HIGH SCHOOL STUDENTS

Princeton

Course II
Lecture
Notes
4 of 7

Princeton SWIM 2010

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Markov Chains

Definition

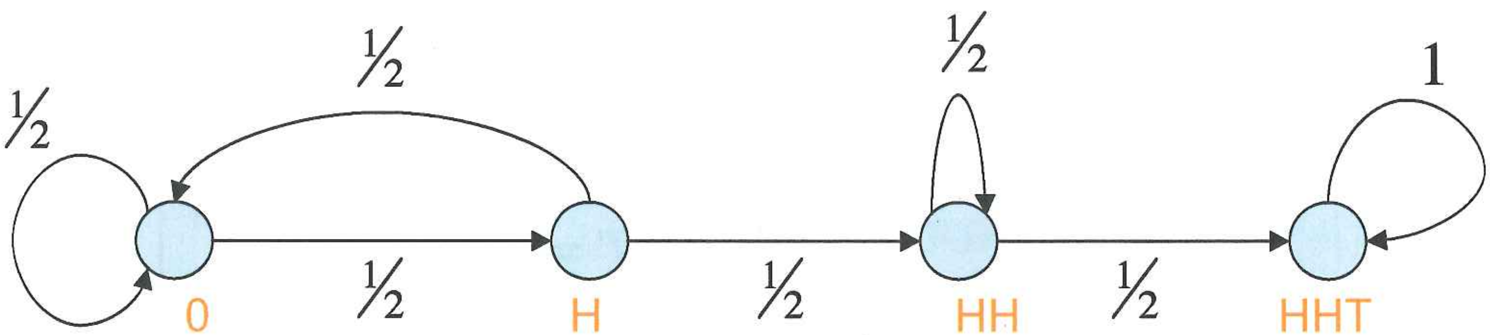
A **Markov process** is a stochastic (random) process with the following property:

The probability of any future behavior of the process depends only on the current state, not on its past behavior. (e.g., Markov property)

Markov Chains

Example - Flipping Coins

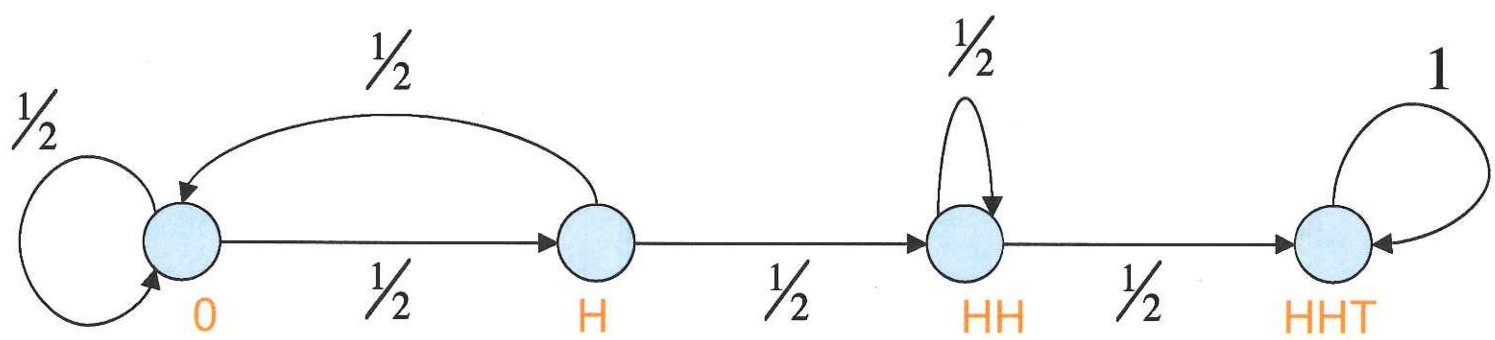
You are going to successively flip a quarter until the pattern HHT appears.



Markov Chains

Example - Flipping Coins

State Diagram



Transition Probability Matrix

Row Stochastic

$$T = \begin{pmatrix} \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 \\ \tfrac{1}{2} & 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Markov Chains

Example - Flipping Coins

Question: On average, how many flips will be required until the pattern HHT appears?

Average number of flips required: $$v = 1 + \beta v$$

$$v = \begin{pmatrix} 8, & 6, & 2, & 0 \end{pmatrix}$$

Markov Chains

Example - Flipping Coins

Question: In the long run, what fraction of time is spent in each state, no matter in which state the chain began at time 0?

THTHHTTHTHTHTHTHHTHTHHTTTHTHTHTHTHH

$$\pi T = \pi$$

Stationary Distribution

The DeGroot Model

Objectives

- To describe how a group of individuals can reach a consensus of beliefs
- To pool the opinions of each individual into a single group consensus
- To form a common probability distribution of the group consensus
- To determine how *central* an individual is in the network

DeGroot, Morris H. 1974. "Reaching a Consensus." *Journal of the American Statistical Association*, Vol. 69, No. 345, pp. 118-121.

The DeGroot Model

Parameters

Consider a group of *n* individuals

- Original opinions - Vector of Probabilities

$$p(0) = \left(p_1(0), \quad p_2(0), \quad \ldots, \quad p_n(0)\right)$$

- Weight placed on others' opinions - Transition Probability Matrix

$$T_{ij} = \begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,n} \\ t_{2,1} & t_{2,2} & \cdots & t_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n,1} & t_{n,2} & \cdots & t_{n,n} \end{pmatrix}$$

Row Stochastic

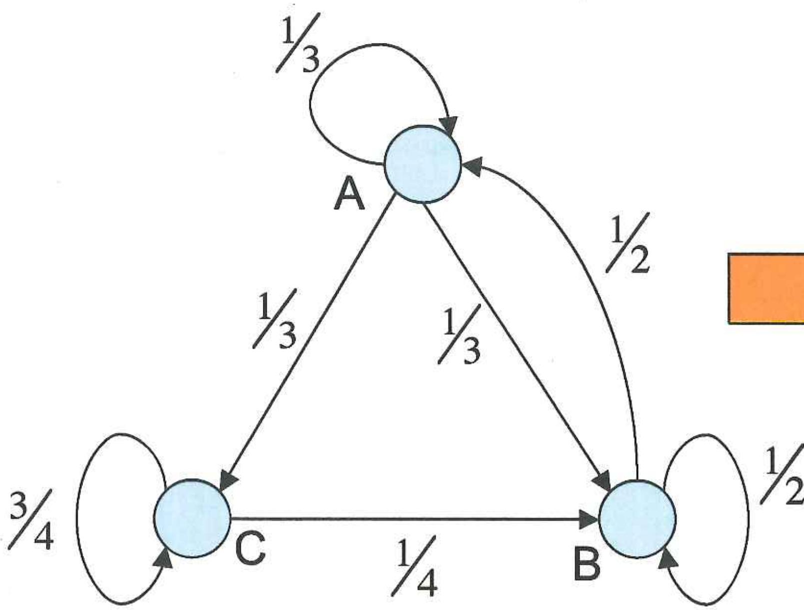
- Updated opinions - Markov Chain

$$p(t) = Tp(t-1) = T^t p(0)$$

The DeGroot Model

Example 1

Suppose there are 3 individuals with the following influence network



$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

The DeGroot Model

Example 1

Suppose the 3 individuals have the following initial vector of beliefs

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The DeGroot Model

Example 1

Updating Beliefs:

$$p(1) = Tp(0) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix}$$

$$p(2) = Tp(1) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/18 \\ 5/12 \\ 1/8 \end{pmatrix}$$

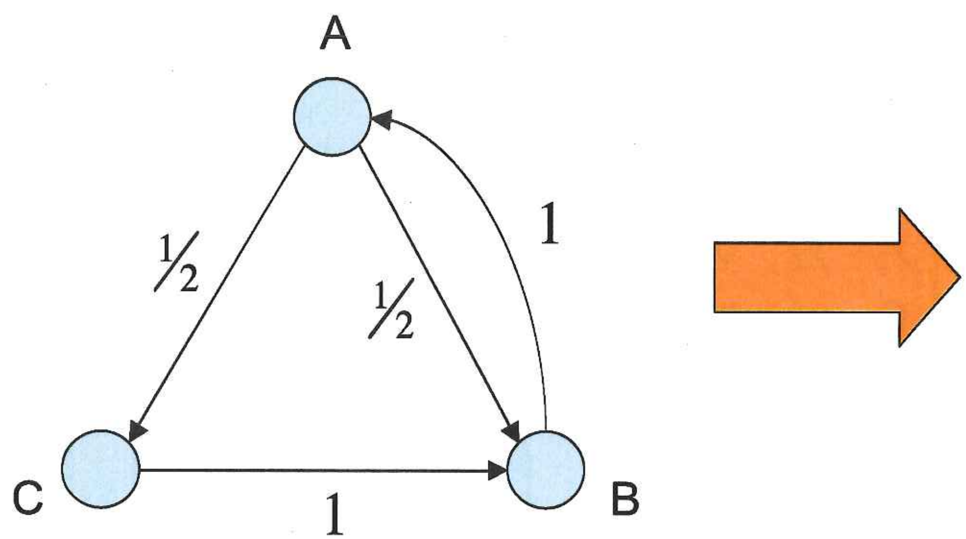
Iteratively,

$$p(t) = Tp(t-1) = T^t p(0) \rightarrow \begin{pmatrix} 3/11 \\ 3/11 \\ 3/11 \end{pmatrix}$$

The DeGroot Model

Example 2

Suppose there are 3 individuals with the following influence network



$$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The DeGroot Model

Example 2

Updating Beliefs:

$$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix},$$

$$T^3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad T^4 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \ldots$$

Iteratively,

$$T^t \to \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

The DeGroot Model

Example 2

$$T^t \to \begin{pmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{pmatrix}$$

$$\lim_{t\to\infty} T^t p(0) = \begin{pmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{pmatrix} \begin{pmatrix} p_1(0) \\ p_2(0) \\ p_3(0) \end{pmatrix}$$

$$p(\infty) = \lim_{t\to\infty} T^t p(0) = 2/5\, p_1(0) + 2/5\, p_2(0) + 1/5\, p_3(0)$$

The DeGroot Model

Example 2

$$p(\infty) = \lim_{t \to \infty} T^t p(0) = \tfrac{2}{5} p_1(0) + \tfrac{2}{5} p_2(0) + \tfrac{1}{5} p_3(0)$$

Stationary Distribution $\pi = \left(\tfrac{2}{5},\ \tfrac{2}{5},\ \tfrac{1}{5}\right)$

Since starting with $p(0)$ or with $p(1) = Tp(0)$ yields the same limit, it must be true that

$$\pi \cdot p(1) = \pi \cdot p(0)$$

$$\pi \cdot \left(Tp(0)\right) = \pi \cdot p(0)$$

⇒ $\pi T = \pi$

The DeGroot Model

Example 2

$$T = \begin{pmatrix} 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T^t \to \begin{pmatrix} \tfrac{2}{5} & \tfrac{2}{5} & \tfrac{1}{5} \\ \tfrac{2}{5} & \tfrac{2}{5} & \tfrac{1}{5} \\ \tfrac{2}{5} & \tfrac{2}{5} & \tfrac{1}{5} \end{pmatrix}$$

$$\pi T = \left(\tfrac{2}{5},\ \ \tfrac{2}{5},\ \ \tfrac{1}{5}\right) \begin{pmatrix} 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \left(\tfrac{2}{5},\ \ \tfrac{2}{5},\ \ \tfrac{1}{5}\right) = \pi$$

The DeGroot Model

Example 2

$$\pi T = (\pi_1, \quad \pi_2, \quad \pi_3)\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (\pi_2, \quad 1/2\pi_1 + \pi_3, \quad 1/2\pi_1) = \pi$$

$$\pi_1 = \pi_2$$

$$\pi_2 = 1/2\pi_1 + \pi_3$$

$$\pi_3 = 1/2\pi_1$$

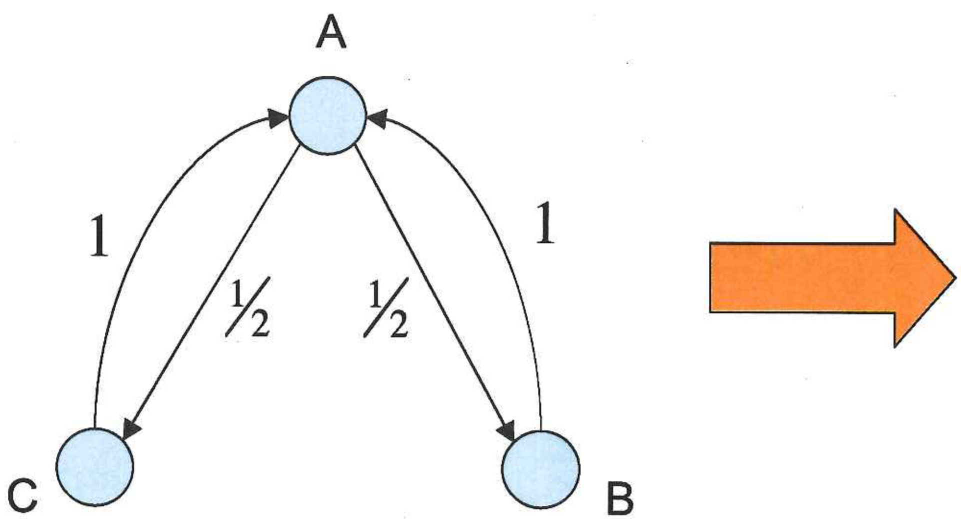
$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\Rightarrow \quad \pi_1 = 2/5, \quad \pi_2 = 2/5, \quad \pi_3 = 1/5$$

The DeGroot Model

Example 3

Suppose there are 3 individuals with the following influence network



$$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The DeGroot Model

Example 3

Updating Beliefs:

$$T = \begin{pmatrix} 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad T^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} \end{pmatrix},$$

$$T^3 = \begin{pmatrix} \tfrac{1}{2} & \tfrac{1}{2} & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad T^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} \end{pmatrix} \ldots$$

No Convergence! Why not?

The DeGroot Model

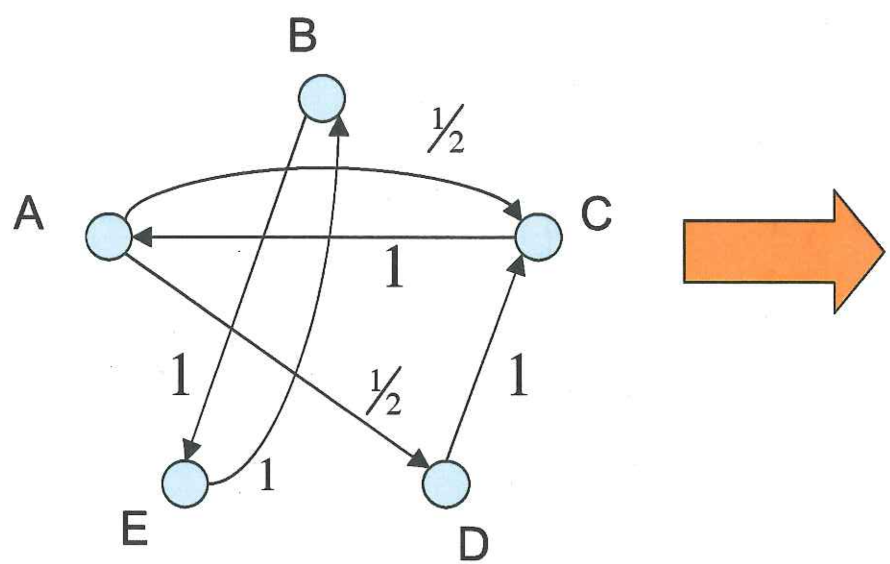
Corollary

Corollary. A consensus is reached in the DeGroot model if and only if the group is *strongly connected* and *aperiodic*.

The DeGroot Model

Example 4

Suppose there are 5 individuals with the following influence network



$$T = \begin{pmatrix} 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The DeGroot Model

Example 4

$$\pi T = \pi$$

$$T = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The DeGroot Model

Implications

Each new opinion depends only on opinion of previous time period

Convergence - *T* is strongly connected (irreducible) and aperiodic

$$\pi T = \pi$$

Stationary Distribution

No Convergence - States of the chain can form at least 2 disjoint closed sets